GALOIS MODULES ARISING FROM FALTINGS'S STRICT MODULES

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Abstract. Suppose O is a complete discrete valuation ring of positive characteristic with perfect residue field. The category of finite flat strict modules was recently introduced by Faltings and appears as an equal characteristic analogue of the classical category of finite flat group schemes. In this paper we obtain a classification of these modules and apply it to prove analogues of properties, which were known earlier for group schemes.

0. Introduction.

Throughout all this paper O is the valuation ring of a complete discrete valuation field K with perfect residue field k of characteristic $p>0$, Γ_K — the absolute Galois group of K and π — a uniformising element of K.

If characteristic of K is 0, denote by $\mathrm{FGr}'(\mathbb{Z}_p)_O$ the category of finite flat commutative group schemes G over O such that the order $|G|$ is a power of p. Any such group scheme appears as a kernel of an isogeny of abelian schemes defined over O and reflects important properties of these abelian schemes.

The classification of objects of the category $\mathrm{FGr}'(\mathbb{Z}_p)_O$ was done in [Fo1] under the restriction $e=1$ on the absolute ramification index $e=e(K)$ of the extension $K/\mathbb{Q}_p$ in terms of finite Honda systems (this classification was not complete for $p=2$, for an improved version cf. [Ab1]). Further progress was done in papers [Ab3] for $e\leqslant p-1$ (group schemes killed by p), [Co] for $e<p-1$ and, finally, in [Br] for an arbitrary e.

Most interesting number theoretic application of the theory of finite flat group schemes come from the study of the structure of the Γ_K-module $H=G(K_{\mathrm{sep}})$ of geometric points of $G\in\mathrm{FGr}'(\mathbb{Z}_p)_O$. We mention the following three results:

A. *Serre's Conjecture* (proved in [Ra]).

This result describes the action of the inertia subgroup $I_K\subset\Gamma_K$ on the semi-simple envelope of H. It is given by characters $\chi: I_K\longrightarrow k^*$ such that for some $N\in\mathbb{N}$, $\chi=\chi_N^a$, where $\chi_N(\tau)=\tau(\pi_N)/\pi_N$, $\pi_N^{p^N-1}=\pi$, and $a=a_0+a_1p+\dots+a_{N-1}p^{N-1}$ with p-digits $a_0,a_1,\dots,a_{N-1}\in[0,e]$;

B. *Ramification estimates.*

If $p^M\,\mathrm{id}_G=0$ then the ramification subgroups $\Gamma_K^{(v)}$ of Γ_K act trivially on H if $v>e(M-1+1/(p-1))$, cf. [Fo2];

C. *Complete description of* Γ_K*-module* H (in the case $e=1$, $p>3$ and $pH=0$);

If $e=1$, $p>3$ and $\mathbb{F}_p[\Gamma_K]$-module H satisfies the above Serre's Conjecture and the ramification estimates (i.e. the ramification subgroups $\Gamma_K^{(v)}$ act trivially on H if $v>\dfrac{1}{p-1}$), then there is an $G\in \mathrm{FGr}'(\mathbb{Z}_p)_O$ such that $H=G(K_{\mathrm{sep}})$, cf.[Ab3].

Suppose now that characteristic of K is p. In this case a reasonable analogue of the concept of finite flat group scheme would give a way to study kernels of isogenies of Drinfeld modules. This analogue should appear as a finite flat commutative group scheme G with continuous action of a closed subring $O_0=\mathbb{F}_q[[\pi_0]]\subset O$. The notion of O_0-module scheme was not very helpful until Faltings introduced in [Fa] a concept of strict O_0-action. The main idea of Faltings's definition can be explained as follows.

Suppose $G=\operatorname{Spec} A$ is a finite flat O_0-module over O. Present A as a quotient $O[X_1,\dots,X_n]/I$ of a ring of polynomials by an ideal I and define a deformation $A^\flat$ as $O[X_1,\dots,X_n]/(I\cdot I_0)$, where $I_0=(X_1,\dots,X_n)$. Faltings requires that O_0-module structure on G, which is given by endomorphisms $[o]:A\longrightarrow A$, $o\in O_0$, should have an extension to an O_0-module structure of the deformation $(A,A^\flat)$ of the algebra A and this extension must satisfy the condition of strictness. This means that if $[o]^\flat:A^\flat\longrightarrow A^\flat$ is an extension of $[o]$ for $o\in O_0$, then $[o]^\flat$ must induce multiplication by o on I_0/I_0^2 and $I/(I\cdot I_0)$. This definition gives the category $\mathrm{FGr}'(O_0)_O$ of strict O_0-modules and its objects have many interesting properties discussed in [Fa].

In this paper we study number theoretic properties of strict O_0-modules. In n.1 we present a concept of strict O_0-module in a slightly different but an equivalent to the original definition by Faltings way. In n.2 we describe the category of strict $\mathbb{F}_q$-modules over O and apply this in n.3 to the classification of objects of the category $\mathrm{FGr}'(O_0)_O$. This classification does not depend on the ramification index of K over $\operatorname{Frac} O_0=K_0$ and requires only the study of primitive elements (i.e. the elements $a\in A(G)$ such that $\Delta a=a\otimes 1+1\otimes a$ where Δ is comultiplication) of the O-algebra $A(G)$ of $G\in\mathrm{FGr}'(O_0)_O$. We apply then this classification to prove that any object of $\mathrm{FGr}'(O_0)_O$ can be embedded into a π_0-divisible group over O. This section contains also a comparison of our antiequivalence with parallel results in the theory of finite flat group schemes and p-adic representations in the mixed characteristic case from papers [Ab1,3], [Br] and [Fo4]. In n.4 we establish precise analogues of the above properties A,B and C in the category $\mathrm{FGr}'(O_0)_O$.

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1. Definition and simplest properties.

Let O be the valuation ring of a complete discrete valuation field K with perfect residue field k of characteristic $p>0$. All O-algebras are assumed usually to be finite, i.e. to be free O-modules of finite rank.

1.1. *Deformations of augmented O-algebras.*

For an augmented O-algebra A, we agree to use the following notation: $\varepsilon_A: A\longrightarrow O$ — the morphism of augmentation; $\operatorname{Ker}\varepsilon_A=I_A$ — the augmentation ideal; $A^{loc}=\varprojlim_n A/I_A^n$ — the I_A-completion of A; $\eta_A:A\longrightarrow A^{loc}$ — a natural projection. Notice, the correspondence $A\mapsto A^{loc}$ is functorial.

The objects of the category DAlg_O are the triples $\mathcal{A} = (A, A^\flat, i_\mathcal{A})$, where A is a finite O-algebra, $A^\flat$ is an O-algebra such that there is a polynomial ring $O[\bar{X}] = O[X_1, \dots, X_n]$, $n \geqslant 0$, and its ideal I such that $A^{loc} = O[\bar{X}]/I$ and $A^\flat = O[\bar{X}]/(I \centerdot I_0)$ with $I_0 := (X_1, \dots, X_n) \supset I$, and $i_\mathcal{A} : A^\flat \longrightarrow A^{loc}$ is a natural epimorphism of O-algebras.

A morphism $\bar{f} = (f, f^\flat) : \mathcal{A} \longrightarrow \mathcal{B} = (B, B^\flat, i_\mathcal{B})$ in DAlg_O is given by a morphism of augmented O-algebras $f : A \to B$ and an O-algebra morphism $f^\flat : A^\flat \longrightarrow B^\flat$ such that $i_\mathcal{A} \circ f^{loc} = f^\flat \circ i_\mathcal{B}$.

In the category DAlg_O, $\mathcal{O} = (O, O, \mathrm{id}_O)$ is an initial object and any $\mathcal{A} = (A, A^\flat, i_\mathcal{A})$ has a natural augmentation to $\mathcal{O}$, $\varepsilon_\mathcal{A} = (\varepsilon_A, \varepsilon_{A^\flat}) : \mathcal{A} \longrightarrow \mathcal{O}$, where $\mathrm{Ker}\, \varepsilon_{A^\flat} = \mathrm{Ann}(\mathrm{Ker}\, i_\mathcal{A}) := I_{A^\flat}$.

Notice that the O-modules $t^*_\mathcal{A} = I_0/I_0^2$ (this module is free) and $N_\mathcal{A} = I/(I \cdot I_0)$ do not depend on the choice of the covering $O[\bar{X}] \longrightarrow A^\flat$. Indeed, the first coincides with $I_{A^\flat}/I^2_{A^\flat}$ and the second — with $\mathrm{Ker}\, i_\mathcal{A}$.

If $\mathcal{A} = (A, A^\flat, i_\mathcal{A})$ and $\mathcal{B} = (B, B^\flat, i_\mathcal{B})$ are objects of DAlg_O and $f : A \longrightarrow B$ is a morphism of augmented O-algebras, then the set of all $f^\flat$ such that $(f, f^\flat) \in \mathrm{Hom}_{\mathrm{DAlg}_O}(\mathcal{A}, \mathcal{B})$ is not empty and has a natural structure of a principal homogeneous space over the group $\mathrm{Hom}_{O\text{-mod}}(t^*_\mathcal{A}, N_\mathcal{B})$.

The deformation $\mathcal{A} = (A, A^\flat, i_\mathcal{A})$ of A will be called minimal if $i_\mathcal{A}$ induces isomorphism of $t^*_\mathcal{A} \otimes k$ onto $I_A/I_A^2 \otimes k$. In other words, f is minimal if minimal systems of generators of O-algebras $A^\flat$ and A^{loc} contain the same number of elements. It is easy to see that if $\mathcal{A}$ is minimal, $\mathcal{A}' = (A, A'^\flat, i_{\mathcal{A}'}) \in \mathrm{DAlg}_O$ and $\bar{f} = (\mathrm{id}_A, f^\flat) \in \mathrm{Hom}_{\mathrm{DAlg}_O}(\mathcal{A}, \mathcal{A}')$ then there is an $\bar{g} \in \mathrm{Hom}_{\mathrm{DAlg}_O}(\mathcal{A}', \mathcal{A})$ such that $\bar{f} \circ \bar{g} = \mathrm{id}_\mathcal{A}$. In particular, all minimal deformations of a given O-algebra A are isomorphic in DAlg_O.

Let DAlg^*_R be a quotient category for DAlg_O: it has the same objects but its morphisms are equivalence classes of morphisms from $\mathrm{Hom}_{\mathrm{DAlg}_O}(\mathcal{A}, \mathcal{B})$ arising from the same O-algebra morphisms $f : A \longrightarrow B$. Then the forgetful functor $\mathcal{A} = (A, A^\flat, i_\mathcal{A}) \mapsto A$ is an equivalence of DAlg^*_O and the category of augmented finite O-algebras.

1.2. *Deformations of affine group schemes.*

Let DSch_O be the dual category for DAlg_O. Its objects appear in the form $\mathcal{H} = \mathrm{Spec}\, \mathcal{A} = (H, H^\flat, i_\mathcal{H})$, where $H = \mathrm{Spec}\, A$ and $H^\flat = \mathrm{Spec}\, A^\flat$ are finite flat O-schemes, $\mathcal{A} = (A, A^\flat, i_\mathcal{A}) \in \mathrm{DAlg}_O$, and $i_\mathcal{H} : H \to H^\flat$ is a closed embedding of O-schemes. This category has direct products: if for $i = 1, 2$, $\mathcal{A}_i = (A_i, A_i^\flat, i_{\mathcal{A}_i})$ with $A_i^{loc} = O[\bar{X}_i]/I$, $A_i^\flat = O[\bar{X}_i]/(I_i \cdot I_{0i})$, then the product $\mathrm{Spec}\, \mathcal{A}_1 \times \mathrm{Spec}\, \mathcal{A}_2$ is given by $\mathrm{Spec}(\mathcal{A}_1 \otimes \mathcal{A}_2)$, where $\mathcal{A}_1 \otimes \mathcal{A}_2 := (A_1 \otimes_O A_2, (A_1 \otimes_O A_2)^\flat, \kappa)$, $(A_1 \otimes_O A_2)^\flat$ is the quotient of $O[\bar{X}_1 \otimes 1, 1 \otimes \bar{X}_2]$ by the product of ideals $I_1 \otimes 1 + 1 \otimes I_2$ and $I_{01} \otimes 1 + 1 \otimes I_{02}$ and κ is the natural projection. Notice that for $i = 1, 2$, two projections pr_i from this product to its components $\mathrm{Spec}\, \mathcal{A}_i$ come from the natural embeddings of $O[\bar{X}_i]$ into $O[\bar{X}_1 \otimes 1, 1 \otimes \bar{X}_2]$.

Let FGr_O be the category of group objects in DSch_R. If $\mathcal{G} = \mathrm{Spec}\, \mathcal{A} \in \mathrm{FGr}_O$ then its group structure is given via comultiplication $\bar{\Delta} = (\Delta, \Delta^\flat) : \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A}$, counit $\bar{\varepsilon} = (\varepsilon, \varepsilon^\flat) : \mathcal{A} \longrightarrow \mathcal{O}$ and coinversion $\bar{\imath} = (i, i^\flat) : \mathcal{A} \longrightarrow \mathcal{A}$ morphisms, which satisfy usual axioms. The morphisms in FGr_O are morphisms of group objects. As usually, FGr_O is an additive category.

Notice that

a) $G = \operatorname{Spec} A$ is a finite flat group scheme over O with comultipilication Δ, counit ε and coinversion i;
b) $\bar{\varepsilon} = \varepsilon_{\mathcal{A}}$, cf. n.1.1.
c) the counit axiom gives for $i = 1, 2$, $\Delta_i^\flat \circ \mathrm{pr}_i = \mathrm{id}_{A^\flat}$ and implies a uniqueness of $\Delta^\flat$ as a lifting of Δ;
d) if $\mathcal{A} = (A, A^\flat, i_{\mathcal{A}}) \in \mathrm{DAlg}_O$ and $G = \operatorname{Spec} A$ is an affine group scheme then there is a unique structure of group object on $\operatorname{Spec} \mathcal{A}$ compatible with that of G;
e) if $f : G \longrightarrow H$ is a morphism of group schemes and $(f, f^\flat) \in \mathrm{Hom}_{\mathrm{DSch}_O}(\mathcal{G}, \mathcal{H})$ then $(f, f^\flat) \in \mathrm{Hom}_{\mathrm{FGr}_O}(\mathcal{G}, \mathcal{H})$.

The above properties have the following interpretation. Define the quotient category FGr_O^* as the category consisiting of the objects of the category FGr_O but where $\mathrm{Hom}_{\mathrm{FGr}_O^*}(\mathcal{G}, \mathcal{H})$ consists of equivalence classes of morphisms from the category FGr_O which induce the same morphisms of affine group schemes $G \to H$. Then the natural functor $\mathcal{G} \mapsto G$ is an equivalence of categories. We can use this equivalence to define the abelian groups of equivalence classes of short exact sequences $\mathrm{Ext}(\mathcal{G}, \mathcal{H})$ in FGr_O. These groups are functorial by both arguments and there are standard 6-terms exact $\mathrm{Hom} - \mathrm{Ext}$-sequences.

1.3. *The categories of strict R-modules.*

Suppose R is a ring and O is an R-algebra.

Suppose $\mathcal{G}$ is an R-module object in the category DSch_O. Then $\mathcal{G}$ is an object of FGr_O and there is a map $R \longrightarrow \mathrm{End}_{\mathrm{FGr}_O}(\mathcal{G})$ satisfying the usual axioms from the definition of R-module. For $r \in R$ and $\mathcal{G} = \operatorname{Spec} \mathcal{A}$, denote by $[\bar{r}] = ([r], [r]^\flat)$ the morphism of action of r on $\mathcal{A} = (A, A^\flat, i_{\mathcal{A}})$. Clearly, $G = \operatorname{Spec} A$ is an R-module in the category of finite flat schemes over O. For any such G, the R-module structure on the deformation $(G, G^\flat, i) \in \mathrm{FGr}_O$ is given by liftings $[r]^\flat : A^\flat \longrightarrow A^\flat$ of morphisms $[r] : A \longrightarrow A$, $r \in R$. Notice that $[r]^\flat$ are morphisms of augmented algebras. All such liftings are automatically compatible with the group structure on this deformation, i.e. for any $r \in R$, it holds $[r] \circ \Delta^\flat = \Delta \circ ([r] \otimes [r])$. So, the above system gives an R-module structure if and only if for any $r_1, r_2 \in R$,

$$[r_1 + r_2]^\flat = \Delta^\flat \circ ([r_1] \otimes [r_2])^\flat, \quad [r_1 r_2]^\flat = [r_1] \circ [r_2]^\flat \tag{1}$$

where $([r_1] \otimes [r_2])^\flat$ is induced by $[r_1]^\flat \otimes [r_2]^\flat$.

We denote by $\mathrm{FGr}(R)_O$ the category of such R-module objects $\mathcal{G}$ where R-action is strict, i.e. if $\mathcal{G} = \operatorname{Spec} \mathcal{A}$ then any $r \in R$ acts on $t_{\mathcal{A}}^*$ and $N_{\mathcal{A}}$ via scalar multiplication by r. This is a basic definition from Faltings' paper [Fa].

Suppose $\mathcal{G}_1 = (G, G_1^\flat, i_1) \in \mathrm{FGr}_O$ and $\mathcal{G}_2 = (G, G_2^\flat, i_2) \in \mathrm{FGr}_O$ are two deformations of a finite flat group scheme G over O. By 1.2, $\mathcal{G}_1$ and $\mathcal{G}_2$ are isomorphic in the category FGr_O^*. Suppose $\mathcal{G}_1$ is equiped with the strict R-action. Then there is a unique (strict) R-action on $\mathcal{G}_2$ such that any $(\mathrm{id}_G, \phi) \in \mathrm{Hom}_{\mathrm{FGr}_O}(\mathcal{G}_1, \mathcal{G}_2)$ and any $(\mathrm{id}_G, \psi) \in \mathrm{Hom}_{\mathrm{FGr}_O}(\mathcal{G}_2, \mathcal{G}_1)$ are, actually, morphisms in the category $\mathrm{FGr}(R)_O$.

Denote by $\mathrm{FGr}^*(R)_O$ the quotient category of $\mathrm{FGr}(R)_O$ where morphisms are equivalence classes of morphisms $(G, G^\flat, i) \longrightarrow (H, H^\flat, j)$ in the category $\mathrm{FGr}(R)_O$ inducing the same morphism $G \longrightarrow H$. By the above property, all isomorphism classes of objects in $\mathrm{FGr}(R)_O$ appear as R-module finite flat schemes G together with a lifting of its R-action to some chosen deformation $G^\flat$ which satisfies the above conditions (1).

For example, if $q=p^n$ with $n\in\mathbb{N}$, the objects of the category $\mathrm{FGr}(\mathbb{F}_q)_O$ appear as $\mathrm{Spec}\,\mathcal{A}$, where $\mathcal{A}=(A,A^\flat,i_{\mathcal{A}})$, $A^{loc}=O[\bar{X}]/I$ and $A^\flat=O[\bar{X}]/(I\cdot I_0)$, and $\mathbb{F}_q$-action is induced by the scalar action of $\mathbb{F}_q$ on $O[\bar{X}]$ (i.e. $[\alpha](\bar{X})=\alpha\bar{X}$, $\alpha\in\mathbb{F}_q$) and there are generators $j_1,\dots,j_n$ of the ideal I such that $[\alpha]j_i=\alpha j_i$ for all $i=1,\dots,n$ and $\alpha\in\mathbb{F}_q$.

If $R=\mathbb{F}_q[\pi]$ with an indeterminate π and $\mathcal{G}\in\mathrm{FGr}(R)_O$ then $\mathcal{G}\in\mathrm{FGr}(\mathbb{F}_q)_O$ and (in addition to the above assumptions) R-action will be determined completely by the action of π given by the correspondence

$$\bar{X}\mapsto[\pi](\bar{X})=\pi\bar{X}+\bar{F}(\bar{X})$$

where $\bar{F}$ is any vector power series with coefficients in O from I_0^2 such that $\bar{F}(\alpha\bar{X})=\alpha\bar{F}(\bar{X})$ for all $\alpha\in\mathbb{F}_q$. This action is strict iff $[\pi]j_i\equiv\pi j_i \bmod(I\cdot I_0)$ for the above generators $j_1,\dots,j_n$ of I.

1.4. Suppose $i:\mathcal{G}_1\longrightarrow\mathcal{G}$ is a closed embedding of strict R-modules. Then the quotient $\mathcal{G}_2=\mathcal{G}/\mathcal{G}_1$ has a natural structure of a strict R-module scheme and the projection $j:\mathcal{G}\longrightarrow\mathcal{G}_2$ is a morphism in the category $\mathrm{FGr}(R)_O$. Similarly, if $j:\mathcal{G}\longrightarrow\mathcal{G}_2$ is a fully faithful morphism of strict R-module schemes then its kernel $\mathcal{G}_1$ is a strict R-module and its (closed) embedding $i:\mathcal{G}_1\longrightarrow\mathcal{G}$ is a morphism from $\mathrm{FGr}(R)_O$.

Notice that for any $\mathcal{G}=(G,G^\flat,i_{\mathcal{G}})\in\mathrm{FGr}(R)_O$, the short exact sequence of group schemes

$$0\longrightarrow G^{\mathrm{loc}}\longrightarrow G\longrightarrow G^{\mathrm{et}}\longrightarrow 0 \tag{2}$$

(here, G^{loc} is the maximal local subgroup scheme of G and G^{et} is the maximal etale quotient of G) induces a short exact sequence in $\mathrm{FGr}(R)_O$

$$0\longrightarrow \mathcal{G}^{\mathrm{loc}}\longrightarrow \mathcal{G}\longrightarrow \mathcal{G}^{\mathrm{et}}\longrightarrow 0 \tag{3}$$

where $\mathcal{G}^{\mathrm{loc}}=(G^{\mathrm{loc}},G^\flat)$, $\mathcal{G}^{\mathrm{et}}=(G^{\mathrm{et}},\mathrm{Spec}\,O)$.

Finally, notice that over the valution ring O_1 of some unramified extension K_1 of K, $G^{\mathrm{et}}\otimes O_1$ is constant, i.e. $A(G^{\mathrm{et}})\otimes O_1=\mathrm{Map}(H^{\mathrm{et}},O_1)$ with $H^{\mathrm{et}}=G^{\mathrm{et}}(K_1)$. This also implies the existence of a finite extension K_1' of K such that the short exact sequence (2) (as well as (3)) splits over the valuation ring of K_1'.

2. Group schemes with strict $\mathbb{F}_q$-action.

As earlier, O is the valuation ring of a complete discrete valuation field K of characteristic p with perfect residue field k, $q=p^{N_0}$ with $N_0\in\mathbb{N}$. In this section we assume that O is an $\mathbb{F}_q$-algebra (then $\mathbb{F}_q\subset k$ and $O\simeq k[[\pi]]$, where π is a uniformiser in K) and study the full subcategory $\mathrm{FGr}'(\mathbb{F}_q)_O$ of the category $\mathrm{FGr}(\mathbb{F}_q)_O$ consisting of strict finite $\mathbb{F}_q$-modules G over O with etale generic fibre. As it was noticed in n.1, its objects appear as finite flat group schemes $G=\mathrm{Spec}\,A(G)$ over O with $\mathbb{F}_q$-action such that $A(G^{loc})$ (or even $A(G)$) can be presented in the form $O[\bar{X}]/I$ in such a way that $\mathbb{F}_q$-action on G is induced by a scalar $\mathbb{F}_q$-action on coordinates of $\bar{X}$ and on some system of generators of the ideal I.

2.1. *The category* $\mathrm{Mod}_{O,\sigma}$ *and the functor* Gr.

Let $\sigma : O \to O$ be the endomorphism of q-th power. For any O-module M, set $M_\sigma = M \otimes_O O$ where an O-module structure on O is given via σ. Objects of the category $\mathrm{Mod}_{O,\sigma}$ are free O-modules M of finite rank with an O-linear morphism $\Phi : M_\sigma \to M$ such that its K-linear extension Φ_K is an isomorphism of K-vector spaces $M_{\sigma,K} = M_\sigma \otimes_O K$ and $M_K = M \otimes_O K$. Morphisms in $\mathrm{Mod}_{O,\sigma}$ are morphisms of O-modules commuting with Φ-action.

For an $M \in \mathrm{Mod}_{O,\sigma}$, introduce the O-algebra

$$A[M] = \mathrm{Sym}_O\, M / \{\Phi m - m^{\otimes q} \mid m \in M\}$$

In other words, if $m_1, \ldots, m_n$ is an O-basis of M and $\Phi m_i = \sum_j o_{ij} m_j$ (where $\det(o_{ij}) \neq 0$), then $A[M] = O[X_1, \ldots, X_n]/(\psi_1, \ldots, \psi_n)$, where $\psi_i = X_i^q - \sum_j o_{ij} X_j$, $i = 1, \ldots, n$. This O-algebra $A[M]$ does not depend on the choice of a basis $m_1, \ldots, m_n$ and gives rise to the strict finite flat $\mathbb{F}_q$-module scheme $G[M] = \mathrm{Spec}\, A[M]$ with comultiplication Δ such that $\Delta(X_i) = X_i \otimes 1 + 1 \otimes X_i$ and an $\mathbb{F}_q$-action such that $[\alpha](X_i) = \alpha X_i$, where $\alpha \in \mathbb{F}_q$, $i = 1, \ldots, n$. The equality $\Phi(M_{\sigma,K}) = M_K$ implies that the generic fibre of $G[M]$ is etale, and therefore, $G[M] \in \mathrm{FGr}'(\mathbb{F}_q)_O$.

Identify M with its image in $A[M]$ via the natural map $m_i \mapsto X_i$, $i = 1, \ldots, n$.

Proposition 1.
a) $M = \{m \in A[M] \mid \Delta m = m \otimes 1 + 1 \otimes m, \quad [\alpha]m = \alpha m, \alpha \in \mathbb{F}_q\}$;
b) *the natural inclusions* $M_i \subset A[M_i]$, $i = 1, 2$, *induce the identification*

$$\mathrm{Hom}_{\mathrm{Mod}_{O,\sigma}}(M_1, M_2) = \mathrm{Hom}_{\mathrm{FGr}(\mathbb{F}_q)_O}(G[M_2], G[M_1]).$$

This proposition implies that the correspondence $M \mapsto G[M]$ induces a fully faithful functor $\mathrm{Gr} : \mathrm{Mod}_{O,\sigma} \longrightarrow \mathrm{FGr}'(\mathbb{F}_q)_O$. The rest of this section will be devoted to the proof of the following theorem.

Theorem 1. *The functor* Gr *is an antiequivalence of categories.*

Notice that the correspondence $\mathrm{Gr} \otimes K : M_K \mapsto G[M](K_{sep})$ is an antiequivalence of the category of σ-etale K-modules and the category of $\mathbb{F}_q[\Gamma_K]$-modules, cf. [Fo4].

2.2. Later we need the following property of the Hochschild cohomology of group schemes $G[M]$.

Lemma 1. *With the above notation suppose that* $a \in A[M]_K$ *is such that* $\delta^+ a := \Delta(a) - a \otimes 1 - 1 \otimes a \in A[M] \otimes A[M]$ *and* $[\alpha]a = \alpha a$ *for all* $\alpha \in \mathbb{F}_q$. *Then there is an* $a' \in A[M]$ *such that* $\delta^+ a = \delta^+ a'$ *and* $a - a' \in M_K$.

Proof of Lemma 1. Let $\delta_k^+ = \delta^+ \otimes k : A[M]_k \longrightarrow A[M]_k^{\otimes 2}$, where $A[M]_k = A[M] \otimes k$. It will be sufficient to prove that

$$\{b \in A[M]_k \mid \delta_k^+ b = 0,\ [\alpha]b = \alpha b,\ \forall \alpha \in \mathbb{F}_q\} = M_k \tag{4}$$

Note that

$$\{X_1^{i_1} \ldots X_n^{i_n} \mid 0 \leqslant i_1, \ldots, i_n < q\}$$

is a k-basis of $A[M]_k$ and

$$\{X_1^{j_1}\dots X_n^{j_n}\otimes X_1^{l_1}\dots X_n^{l_n}\mid 0\leqslant j_1,\dots,j_n,l_1,\dots,l_n<q\}$$

is a k-basis of $A[M]_k^{\otimes 2}$. Now the equality (4) is implied by the following 3 observations:

a) for any $0\leqslant i_1,\dots,i_n<q$, $\delta_k^+(X_1^{i_1}\dots X_n^{i_n})$ is a linear combination of $X_1^{j_1}\dots X_n^{j_n}\otimes X_1^{l_1}\dots X_n^{l_n}$ with $j_1,\dots,j_n,l_1,\dots,l_n\geqslant 0$ such that $j_1+l_1=i_1$, $\dots,j_n+l_n=i_n$;

b) if $\sum\limits_{0\leqslant i_1,\dots,i_n<q}\alpha_{i_1\dots i_n}X_1^{i_1}\dots X_n^{i_n}\in\operatorname{Ker}\delta_k^+$ then for each multi index $(i_1,\dots,i_n)$, either $\alpha_{i_1\dots i_n}=0$, or $\delta_k^+(X_1^{i_1}\dots X_n^{i_n})=0$;

c) for any $0\leqslant i_1,\dots,i_n<q$, $\delta_k^+(X_1^{i_1}\dots X_n^{i_n})=0$ iff
$X_1^{i_1}\dots X_n^{i_n}\in\{X_s^{p^t}\mid 1\leqslant s\leqslant n, 0\leqslant t<N_0\}$.

2.3. For $G\in\mathrm{FGr}(\mathbb{F}_q)_O$, let

$$L(G)=\{a\in A(G)\mid \Delta a=a\otimes 1+1\otimes a,\ [\alpha]a=\alpha a,\forall\alpha\in\mathbb{F}_q\}.$$

Clearly, the above lemma implies that if $M\in\mathrm{Mod}_{O,\sigma}$, then $L(G[M])=M$.

The following lemma can be proved directly from the above definition of $L(G)$.

Lemma 2.
a) *If O_1 is a complete discrete valuation ring containing O, then*
$L(G\otimes O_1)=L(G)\otimes O_1$;
b) *For any two objects G_1 and G_2 of* $\mathrm{FGr}(\mathbb{F}_q)_O$*, it holds*
$L(G_1\times G_2)=L(G_1)\oplus L(G_2)$.

The q-th power map on $A(G)$ induces $\Phi: L(G)_\sigma\longrightarrow L(G)$ which provides $L(G)$ with the structure of an object of the category $\mathrm{Mod}_{O,\sigma}$. Indeed, if $H=G(K_{\mathrm{sep}})$ with the natural structure of $\mathbb{F}_q[\Gamma_K]$-module, then $L(G)_K=\mathrm{Hom}_{\mathbb{F}_q[\Gamma_K]\text{-mod}}(H,K_{\mathrm{sep}})$ and

$$\operatorname{Im}\Phi_K=\mathrm{Hom}_{\mathbb{F}_q[\Gamma_K]\text{-mod}}(H,\sigma(K_{\mathrm{sep}}))\otimes_{\sigma(K)}K=L(G)_K$$

because $\sigma(K_{\mathrm{sep}})\otimes_{\sigma(K)}K=K_{\mathrm{sep}}$.

The correspondence $L: G\mapsto L(G)$ is the functor from $\mathrm{FGr}'(\mathbb{F}_q)_O$ to $\mathrm{Mod}_{O,\sigma}$. Clearly, the composition $\mathrm{Gr}\circ L$ is equivalent to the identity functor on $\mathrm{Mod}_{O,\sigma}$. So, Gr is an antiequivalence (with the quasi-inverse functor L) if any $G\in\mathrm{FGr}'(\mathbb{F}_q)_O$ is isomorphic to $G[M]$ for a suitable $M\in\mathrm{Mod}_{O,\sigma}$. When proving this property we can enlarge (if necessary) the basic ring O by the following lemma.

Lemma 3. *If O' is a complete discrete valuation ring containing O and $G\in$* $\mathrm{FGr}'(\mathbb{F}_q)_O$ *then $G\simeq G[M]$ with $M\in\mathrm{Mod}_{O,\sigma}$ if and only if $G\otimes O_1\simeq G[M_1]$ for some $M_1\in\mathrm{Mod}_{O_1,\sigma}$.*

Proof. Clearly, if $G\simeq G[M]$ with $M\in\mathrm{Mod}_{O,\sigma}$, then $G\otimes O_1\simeq G[M\otimes O_1]$.

If $G\otimes O_1\simeq G[M_1]$ with $M_1\in\mathrm{Mod}_{O,\sigma}$, then $M_1\simeq L(G\otimes O_1)=L(G)\otimes O_1$, therefore, $G\otimes O_1\simeq G[L(G)]\otimes O_1$ and the natural inclusion $G[L(G)]\subset G$ is an isomorphism. Lemma 3 is proved

2.4. *$\mathbb{F}_q$-module schemes of order q.*

Proposition 2. *Suppose G is an $\mathbb{F}_q$-module scheme of order q such that $G\otimes K$ is etale and $\mathbb{F}_q$ acts on the cotangent space t_G^* via scalar multiplication. Then $G=G[M]$, where $M=Om$ and $\Phi m=\lambda m$ with some $\lambda\in O$, $\lambda\neq 0$ (in particular, G is a strict $\mathbb{F}_q$-module).*

Proof. Let $I_G=\operatorname{Ker} e_G$ (where e_G is the counit morphism). If $I_G=I_G^2$ then G is etale and over some bigger (unramified over O) basic ring O_1, $G\otimes O_1$ is constant, i.e. $G\otimes O_1\simeq G[M_1]$, where $M_1=O_1m_1$ and $\Phi(m_1)=m_1$. Therefore, $G\simeq G[M]$ where $M\otimes O_1\simeq M_1$, i.e. $M=Om$, $m=\xi m_1$ with $\xi\in O_1^*$ and $\Phi m=\lambda m$ with $\lambda=\xi^{q-1}\in O\cap O_1^*=O^*$.

Suppose now that $I_G\neq I_G^2$. Consider the decomposition $I_G=\oplus_{1\leqslant n<q}I_{G,n}$, where an $\alpha\in\mathbb{F}_q$ acts on $I_{G,n}$ via multiplication by α^n.

Because $t_G^*=I_G/I_G^2\neq 0$, $\operatorname{rk}_O I_{G,1}\geqslant 1$ and, therefore, for all $1\leqslant n<q$, $\operatorname{rk}_O I_{G,n}\geqslant 1$ (if $x\in I_{G,1}$, $x\neq 0$, then $x^n\neq 0$ and $x^n\in I_{G,n}$). So, all these ranks are equal to 1 and, if $I_{G,1}=xO$, then the equality $I_G=I_{G,1}+I_G^2$ implies for all $1\leqslant n<q$, $I_{G,n}=x^nO$.

Finally, $x^q\in I_{G,1}$ and $x^q=\lambda x$ for some $\lambda\in O$. Clearly, $\lambda\neq 0$ because $G\otimes K$ is etale. This proves that $A(G)=A[M]$ with $M=Om$, $\Phi m=\lambda m$.

Suppose $\Delta x=x\otimes 1+1\otimes x+\delta$. Then $\delta^q=\lambda\delta$ and $\delta\in(x\otimes x)A(G)\otimes A(G)$. This implies easily that $\delta=0$. It remains to note that by the choice of x, $[\alpha]x=\alpha x$ for any $\alpha\in\mathbb{F}_q$, and this action of $\mathbb{F}_q$ is strict.

Definition. We shall denote by μ_λ the $\mathbb{F}_q$-module scheme $G[M]$ from the above proposition.

2.5. Suppose $G\in\mathrm{FGr}'(\mathbb{F}_q)_O$ satisfies the following four assumptions:

1) $G=G^{loc}$;

2) there is a short exact sequence in the category $\mathrm{FGr}'(\mathbb{F}_q)_O$

$$0\longrightarrow\mu_\lambda\xrightarrow{i}G\longrightarrow G_1\longrightarrow 0$$

where $\lambda\in O,\lambda\neq 0$;

3) the above exact sequence splits over K, i.e. there is a Γ_K-invariant section of the projection of $\mathbb{F}_q[\Gamma_K]$-modules $G(K_{sep})\longrightarrow G_1(K_{sep})$;

4) there is an $M_1\in\mathrm{Mod}_{O,\sigma}$ such that $G_1\simeq G[M_1]$.

Proposition 3. *With the above assumptions 1)-4), there is an $M\in\mathrm{Mod}_{O,\sigma}$ such that $G\simeq G[M]$.*

Proof. Let $A(\mu_\lambda)=O[u]$ with $u^q=\lambda u$, $\Delta u=u\otimes 1+1\otimes u$, $[\alpha]u=\alpha u$ for $\alpha\in\mathbb{F}_q$. Suppose the closed embedding $i:\mu_\lambda\longrightarrow G$ is given by the O-algebra morphism $i^*:A\longrightarrow A(\mu_\lambda)$. Set $A=A(G)$ and $B=A(G_1)$. Then

$$s=\Delta_G\circ(i^*\otimes\mathrm{id}_A):A\longrightarrow A(\mu_\lambda)\otimes A$$

is the coaction of μ_λ on A and $B=A^{\mu_\lambda}=\{a\in A\mid s(a)=1\otimes a\}$.

Choose an $a_0\in A$ such that $i^*(a_0)=u^{q-1}$ and $[\alpha]a_0=a_0$ for all $\alpha\in\mathbb{F}_q$. Set

$$s(a_0)=1\otimes a_0+u\otimes a_1+\dots+u^{q-1}\otimes a_{q-1}$$

where $a_1,\dots,a_{q-1}\in A$. Then for any $\alpha\in\mathbb{F}_q$ and $m=1,\dots,q-1$, $[\alpha]a_m=\alpha^{-m}a_m$ and the identity

$$(\mathrm{id}\otimes i^*)(s(a_0))=\Delta(i^*a_0)=\Delta(u^{q-1})=(u\otimes 1+1\otimes u)^{q-1}$$

implies that $i^*(a_{q-1}) = 1$ and $i^*(a_{q-2}) = -u$. Therefore, $a_{q-1} \in A^*$ (because $A = A^{loc}$) and $A = B[a_{q-2}]$ (because a_{q-2} generates A modulo a nilpotent ideal $I_B A$).

The equality

$$1 \otimes s(a_0) + u \otimes s(a_1) + \cdots + u^{q-1} \otimes s(a_{q-1}) = (\mathrm{id} \otimes s)(s(a_0))$$

$$= (\Delta \otimes \mathrm{id})(s(a_0)) = 1 \otimes 1 \otimes a_0 + (u \otimes 1 + 1 \otimes u) \otimes a_1 + \cdots + (u \otimes 1 + 1 \otimes u)^{q-1} \otimes a_{q-1}$$

implies that $s(a_{q-1}) = 1 \otimes a_{q-1}$ and $s(a_{q-2}) = 1 \otimes a_{q-2} - u \otimes a_{q-1}$.

Therefore, $a_{q-1} \in B \cap A^* = B^*$ and for $\theta = -a_{q-2}/a_{q-1}$, we have $A = B[\theta]$, $s(\theta) = 1 \otimes \theta + u \otimes 1$, $[\alpha]\theta = \alpha\theta$ for $\alpha \in \mathbb{F}_q$, and $\theta^q - \lambda\theta = b \in B$.

Notice also that $\delta^+\theta = \Delta\theta - \theta \otimes 1 - 1 \otimes \theta \in B \otimes B$, because this is a $(\mu_\lambda \times \mu_\lambda)$-invariant element of $A \otimes A$.

From condition 3) it follows the existence of $v \in A_K$ such that $i^*(v) = u$, $v^q = \lambda v$, $\Delta v = v \otimes 1 + 1 \otimes v$ and $[\alpha]v = \alpha v$ for $\alpha \in \mathbb{F}_q$. Therefore, denoting the K-linear extension of s by s_K, we have $s_K(v) = 1 \otimes v + u \otimes 1$. This implies that $\theta - v = b_0 \in B \otimes K$, and $\delta^+ b_0 = \delta^+\theta$.

Therefore, by the property of Hochshild cohomology from the end of n.2.1, there is a $b_1 \in B$ (with the property $[\alpha]b_1 = \alpha b_1$, $\forall \alpha \in \mathbb{F}_q$) such that $\delta^+ b_1 = \delta^+\theta$ and replacing θ by $\theta - b_1$ we can assume that $\Delta\theta = \theta \otimes 1 + 1 \otimes \theta$. Therefore, $M = M_1 + O\theta \in \mathrm{Mod}_{O,\sigma}$, $A = A[M]$ and $G = G[M]$. The proposition is proved.

2.6. Now we can finish the proof of Theorem 1.

Suppose $G \in \mathrm{FGr}'(\mathbb{F}_q)_O$ is of order q^N, $N \in \mathbb{N}$. As it was noticed in n.2.3, it will be sufficient to prove that $G \simeq G[M]$ for a suitable $M \in \mathrm{Mod}_{O,\sigma}$. When proving this property we can enlarge (if necessary) the basic ring O.

Apply induction on N.

The case $N = 1$ follows from Proposition 2 in n.2.4.

Suppose that $G \in \mathrm{FGr}'(\mathbb{F}_q)_O$ is of order q^N and any $\mathbb{F}_q$-module scheme of order q^{N-1} appears in the form $G[M_1]$ with $M_1 \in \mathrm{Mod}_{O,\sigma}$. By n.1.4 and Lemma 2b), it will be sufficient to consider the following two cases:

a) $G = G^{\mathrm{et}}$;

In this case we can enlarge O to assume that G is a constant group scheme, where the property $G \simeq G[M]$ with $M \in \mathrm{Mod}_{O,\sigma}$ is obviously true.

b) $G = G^{\mathrm{loc}}$.

In this case enlarge O to be able to assume that $G \otimes K$ is constant, i.e. $H = G(K_{\mathrm{sep}})$ has a trivial Γ_K-action. Choose a 1-dimensional $\mathbb{F}_q$-subspace H_2 in H. It gives rise to a short exact sequence of $\mathbb{F}_q$-module group schemes

$$0 \longrightarrow G_2 \longrightarrow G \longrightarrow G_1 \longrightarrow 0$$

where $G_2(K_{\mathrm{sep}}) = H_2$.

Because G is a strict $\mathbb{F}_q$-module, $\mathbb{F}_q$ acts on t_G^* and, therefore, on $t_{G_2}^*$, via scalar multiplication. By Proposition 2, $G_2 \simeq \mu_\lambda$, $\lambda \in O, \lambda \neq 0$, belongs to $\mathrm{FGr}'(\mathbb{F}_q)_O$ and, therefore, $G_1 \in \mathrm{FGr}'(\mathbb{F}_q)_O$, cf. n.1.4. By induction, we have $G_1 \simeq G[M_1]$, $M_1 \in \mathrm{Mod}_{O,\sigma}$.

Finally, G satisfies the properties 1)-4) from n.2.5 and by Proposition 3, $G \simeq G[M]$ for $M \in \mathrm{Mod}_{O,\sigma}$. The Theorem is proved.

3. Group schemes with strict O_0-action.

In this section $R = O_0$ is the valuation ring of a closed subfield K_0 of K with finite residue field $\mathbb{F}_q$, $q = p^n$, $n \in \mathbb{N}$. Fix a choice of uniformising element π_0 in K_0. So, $O_0 = \mathbb{F}_q[[\pi_0]]$.

3.1. Denote by $\mathrm{FGr}'(O_0)_O$ a full subcategory in $\mathrm{FGr}(O_0)_O$ consisting of objects killed by some power of $[\pi_0]$. This is an analogue of the classical category of p-torsion finite flat group schemes over the valuation ring of a complete discrete valuation field of mixed characteristic.

Let $\mathrm{Mod}(O_0)_O$ be the category, consisting of $M \in \mathrm{Mod}_{O,\sigma}$ with O_0-module structure given for $o \in O_0$ by endomorphisms $[o] \in \mathrm{End}_{\mathrm{Mod}_{O,\sigma}}(M)$ such that for all $m \in M$, $[\alpha]m = \alpha m$ if $\alpha \in \mathbb{F}_q \subset k$ and $[\pi_0^N]m = 0$ for sufficiently large N. Morphisms in the category $\mathrm{Mod}(O_0)_O$ are morphisms of the category $\mathrm{Mod}_{O,\sigma}$ which commute with all endomorphisms $[o]$, $o \in O_0$. In other words, any $M \in \mathrm{Mod}(O_0)_O$ is a free O-module M of finite rank equiped with

a) an O-linear map $\Phi : M_\sigma \longrightarrow M$ such that $\Phi \otimes K$ is isomorphism;

b) an O-linear nilpotent endomorphism $[\pi_0] : M \longrightarrow M$ commuting with Φ.

We shall denote by $\mathrm{Mod}'(O_0)_O$ a full subcategory of $\mathrm{Mod}(O_0)_O$ consisting of objects M such that

c) for any $m \in M$, $[\pi_0]m \equiv \pi_0 m \bmod \Phi(M_\sigma)$.

The category $\mathrm{Mod}(O_0)_O$ is not generally abelian. But similarly to the category $\mathrm{FGr}'(O_0)_O$, one can work with short exact sequences in $\mathrm{Mod}(O_0)_O$. The sequence of objects and morphisms in $\mathrm{Mod}(O_0)_O$

$$0 \longrightarrow M_1 \xrightarrow{i} M \xrightarrow{j} M_2 \longrightarrow 0$$

is exact if and only if i is a pure morphism of O-modules (i.e. $M/i(M_1)$ has no O-torsion) and j is epimorphism of O-modules with kernel $i(M_1)$. This allows to define O-modules of equivalence classes of short exact sequences which satisfy the usual functorial properties.

3.2. Let $i : \mathrm{FGr}(O_0)_O \longrightarrow \mathrm{FGr}(\mathbb{F}_q)_O$ be the forgetful functor.

Proposition 4. *If $\mathcal{G} \in \mathrm{FGr}'(O_0)_O$, then $i(\mathcal{G}) \in \mathrm{FGr}'(\mathbb{F}_q)_O$.*

Proof. Indeed, if $\mathcal{G} = (G, G^\flat)$ then O_0 acts via a scalar multiplication on $t^*_{G^\flat}$. This implies that t^*_G is killed by some power of π_0 and, therefore, $t^*_{G\otimes K} = t^*_G \otimes K = 0$, i.e. $G \otimes K$ is etale. The proposition is proved.

Let $L : \mathrm{FGr}'(\mathbb{F}_q)_O \longrightarrow \mathrm{Mod}_{O,\sigma}$ be the functor from n.2. For any $\mathcal{G} \in \mathrm{FGr}'(O_0)_O$, $L(i(\mathcal{G}))$ has a natural structure of an object of the category $\mathrm{Mod}(O_0)_O$. We denote by L the functor from $\mathrm{FGr}'(O_0)_O$ to $\mathrm{Mod}(O_0)_O$ induced by the above correspondence $\mathcal{G} \mapsto L(i(\mathcal{G}))$.

Theorem 2. *The functor L induces antiequivalence of the categories* $\mathrm{FGr}'(O_0)_O$ *and* $\mathrm{Mod}'(O_0)_O$.

Proof. Let $\mathcal{G} \in \mathrm{FGr}'(O_0)_O$ and $M = L(\mathcal{G}) \in \mathrm{Mod}(O_0)_O$. Considering M as an object of $\mathrm{Mod}_{O,\sigma}$ we can recover the $\mathbb{F}_q$-module scheme $i(\mathcal{G})$ in the form $G[M]$. This reduces the proof of our theorem to the following statement:

Proposition 5. *$\mathcal{G}$ is a strict O_0-module iff M satisfies the condition* c) from n.3.1.

Let $\mathcal{G}^{loc}$ be the maximal local subobject in $\mathcal{G}$ and $M^{loc} = L(\mathcal{G}^{loc})$. Then the natural projection $M \longrightarrow M^{loc}$ induces isomorphism of (O_0, O)-modules $M/\Phi(M_\sigma)$ and $M^{loc}/\Phi(M_\sigma^{loc})$. So, it will be sufficient to prove the above proposition for a local $\mathcal{G}$. Under this assumption choose a vector column $\bar{m} = (m_1, \ldots, m_n)^t$ such that its coordinates create an O-basis of $M = L(\mathcal{G})$ and set $\Phi\bar{m} = C\bar{m}$ and $[\pi_0]\bar{m} = D\bar{m}$, where $C, D \in \mathbb{M}_n(O)$ are $n \times n$-matrices with coefficients from O.

Lemma 4. *The above matrices C and D give a structure of an object of the category* $\mathrm{Mod}'(O_0)_O$ *on M if and only if*

1) $\det C \neq 0$*;*

2) *D is nilpotent and $\sigma(D)C = CD$;*

3) *there is a $B \in \mathbb{M}_n(O)$ such that $D - \pi_0 E_n = BC$.*

Proof. Indeed, 1) means that $\Phi(M_\sigma)$ is a lattice in the O-module M.

Property 2) means that M is $[\pi_0]$-torsion and the morphisms Φ and $[\pi_0]$ commute one with another. Indeed, $(\Phi \circ [\pi_0])(\bar{m}) = [\pi_0](\Phi\bar{m}) = CD\bar{m}$ and $([\pi] \circ \Phi)(\bar{m}) = \sigma(D)C\bar{m}$.

Notice that the coordinates of the vector $C\bar{m}$ generate the O-module $\Phi(M_\sigma)$. So, the coordinates of $[\pi_0](\bar{m}) - \pi_0\bar{m}$ belong to $\Phi(M_\sigma)$ if and only if there is an $B \in \mathbb{M}_n(O)$ such that $D - \pi_0 E_n = BC$.

The lemma is proved.

Remark. a) $M = M^{loc}$ if and only if C is σ-nilpotent, i.e. there is an $N \in \mathbb{N}$ such that $\sigma^N(C) \ldots \sigma(C)C = 0$;
b) the above properties 1) and 3) imply that $\sigma(D) - \pi_0 E_n = CB$.

With the above notation and agreements let $\mathcal{G} = (G, G^\flat)$, where $A(G) = O[\bar{X}]/I$ and $A(G^\flat) = O[\bar{X}]/(I \cdot I_0)$, where $\bar{X} = (X_1, \ldots, X_n)$ and the ideal I is generated by the coordinates of the vector $\bar{X}^q - C\bar{X}$.

Then $A(G)$ has an O-basis $\{X_1^{a_1} \ldots X_n^{a_n} \mid 0 \leqslant a_1, \ldots, a_n < q\}$ and this basis can be completed to an O-basis of $A(G^\flat)$ by joining the elements of the set $\{X_i^q \mid 1 \leqslant i \leqslant n\}$.

The action of O_0 is given by $[\pi_0]\bar{X} = D\bar{X}$ on $A(G)$. This action is strict if and only if it can be extended to $A(G^\flat)$ via the relation

$$[\pi_0]\bar{X} = D\bar{X} + B(\bar{X}^q - C\bar{X})$$

with some $B \in \mathbb{M}_n(O)$ in such a way that it induces the scalar multiplication by π_0 on $t^*_{A(G^\flat)}$ and $N_{A(G^\flat)}$.

The first condition is equivalent to the equality $D - BC = \pi_0 E_n$.

The second can be analysed as follows,

$$[\pi_0]^\flat(\bar{X}^q - C\bar{X}) \equiv ([\pi_0]^\flat\bar{X})^q - C([\pi_0]^\flat\bar{X}) \equiv (\sigma(D) - CB)(\bar{X}^q - C\bar{X})$$

and is equivalent to the matrix equality $\sigma D - CB = \pi_0 E_n$.

So, Proposition 5 and Theorem 2 follow from Lemma 4 and the above Remark b).

3.3. Clearly, the antiequivalence L transforms short exact sequences in $\mathrm{FGr}'(O_0)_O$ to short exact sequences in the category $\mathrm{Mod}'(O_0)_{O,\sigma}$. In particular, we have the following property.

Proposition 6. *Suppose $\mathcal{G}_1, \mathcal{G}_2 \in \mathrm{FGr}'(O_0)_O$ and $f \in \mathrm{Hom}_{\mathrm{FGr}'(O_0)_O}(\mathcal{G}_1, \mathcal{G}_2)$. Then*
a) *f is a closed embedding if and only if $L(f)$ is surjective;*
b) *f is fully faithful if and only if $L(f)$ is a pure embedding.*

Suppose for $n \in \mathbb{N}$, $\mathcal{G}^{(n)} \in \mathrm{FGr}'(O_0)_O$, $i_n : \mathcal{G}^{(n)} \longrightarrow \mathcal{G}^{(n+1)}$ is a closed immersion, $j_n : \mathcal{G}^{(n+1)} \longrightarrow \mathcal{G}^{(n)}$ is a fully faithful morphism. For $m > n$, set $i_{nm} = i_n \circ \cdots \circ i_{m-1}$ and $j_{mn} = j_{m-1} \circ \cdots \circ j_n$. Then (following the original deinition of Tate) $\{\mathcal{G}^{(n)}, i_n, j_n\}$ is a π_0-divisible group in the category $\mathrm{FGr}'(O_0)_O$ if for any $m > n$, $(\mathcal{G}^{(n)}, i_{nm}) = \mathrm{Ker}[\pi_0^m]\,\mathrm{id}_{\mathcal{G}^{(m)}}$ and $[\pi_0^{m-n}]\,\mathrm{id}_{\mathcal{G}^{(m)}} = j_{mn} \circ i_{nm}$.

Similar definitions can be done for the category $\mathrm{Mod}'(O_0)_O$, where a π_0-divisible group appears as the collection $\{M^{(n)}, i_n, j_n\}$ where i_n is a pure embedding of underlying O-modules $M^{(n)} \longrightarrow M^{(n+1)}$ and $j_n : M^{(n+1)} \longrightarrow M^{(n)}$ is a surjection of O-modules.

3.4. Clearly, the functor L from Theorem 2 transforms π_0-divisible groups in the category $\mathrm{FGr}'(O_0)_O$ to π_0-divisible groups in the category $\mathrm{Mod}'(O_0)_O$.

Theorem 3. *For any object $\mathcal{G}$ of the category $\mathrm{FGr}'(O_0)_O$, there is a π_0-divisible group $\{\mathcal{H}^{(n)}, i_n, j_n\}_{n\geqslant 1}$ and a closed embedding $\mathcal{G} \longrightarrow \mathcal{H}^{(N)}$, where $N \in \mathbb{N}$ is such that $[\pi_0^N]\,\mathrm{id}_{\mathcal{G}} = 0$.*

Remark. The statement of the above theorem is equivalent to the existence of a π_0-divisible group $\{\mathcal{H}'^{(n)}, i_n, j_n\}_{n\geqslant 1}$ and a fully faithful morphism $\mathcal{H}'^{(N)} \longrightarrow \mathcal{G}$.

Proof. The antiequivalence L allows us to prove the dual version of this theorem in the category $\mathrm{Mod}'(O_0)_O$.

Let $M \in \mathrm{Mod}'(O_0)_O$ and let $N \in \mathbb{N}$ be such that $[\pi_0^N]M = 0$. Use induction on N.

Suppose first, that $N = 1$.

Then M is a free O-module with σ-linear $\Phi : M_\sigma \longrightarrow M$ such that $\pi_0 M \subset \Phi(M_\sigma) \subset M$. Choose an O-basis $m_1, \dots, m_n$ in M and take a vector column $\bar{m} = (m_1, \dots, m_n)^t$. Then $\Phi\bar{m} = C\bar{m}$, where $C \in \mathbb{M}_n(O)$ is a divisor of $\pi_0 E_n$. Let $\widetilde{C} \in \mathbb{M}_n(O)$ be such that $C\widetilde{C} = \pi_0 E_n$.

For $l \in \mathbb{N}$, introduce free O-modules $M^{(l)}$ with free generators

$$\{m_{1i}^{(k)}, m_{2i}^{(k)} \mid 1 \leqslant i \leqslant n, 1 \leqslant k \leqslant l\}$$

Let $\bar{m}_1^{(0)} = \bar{m}_2^{(0)} = \bar{0}$ and for $k \geqslant 1$, let $\bar{m}_1^{(k)} = (m_{11}^{(k)}, \dots, m_{1n}^{(k)})^t$ and $\bar{m}_2^{(k)} = (m_{21}^{(k)}, \dots, m_{2n}^{(k)})^t$. Define O-linear morphisms $\Phi : M_\sigma^{(l)} \longrightarrow M^{(l)}$ and $[\pi_0] : M \longrightarrow M$ by setting for $1 \leqslant k \leqslant l$,

$$\Phi\bar{m}_1^{(k)} = \widetilde{C}\bar{m}_1^{(k)} + \bar{m}_2^{(k-1)}, \quad \Phi\bar{m}_2^{(k)} = C\bar{m}_2^{(k)} + \bar{m}_1^{(k)}$$

and $[\pi_o]\bar{m}_1^{(k)} = \bar{m}_1^{(k-1)}$, $[\pi_0]\bar{m}_2^{(k)} = \bar{m}_2^{(k-1)}$.

It is easy to see that we defined a structure of objects of the category $\mathrm{Mod}'(O_0)_O$ on all $M^{(l)}$, $l \in \mathbb{N}$, and the system $\{M^{(l)}\}_{l\geqslant 1}$ together with the natural inclusions $i_n : M^{(n} \longrightarrow M^{(n+1)}$ and projections $j_l : M^{(l+1)} \longrightarrow M^{(l)}$ gives a π_0-divisible group in the category $\mathrm{Mod}'(O_0)_O$. Clearly, the correspondences $\bar{m}_1^{(1)} \mapsto \bar{0}$ and $\bar{m}_2^{(1)} \mapsto \bar{m}$ give an epimorphic map from $M^{(1)}$ to M. The case $N = 1$ has been considered.

Suppose $N > 1$ and Theorem 3 has been proved for all $M' \in \mathrm{Mod}'(O_0)_O$ such that $[\pi_0]^{N-1}(M') = 0$.

Let $M_1 = \mathrm{Ker}[\pi_0]^{N-1}\,\mathrm{id}_M$ and $M_2 = [\pi_0]^{N-1}(M) \subset M$. Then M_1 and M_2 have natural structures of objects of the category $\mathrm{Mod}'(O_0)_O$ and in this category we have a natural short exact sequence

$$\varepsilon: \quad 0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0$$

(Notice that, generally, the embedding $M_2 \subset M$ is not pure). By induction, there is a π_0-divisible group $\{T^{(n)}, i_n, j_n\}_{n\geqslant 1}$ and pure embeddings $\alpha : M_1 \longrightarrow T^{(N-1)}$, $\beta : M_2 \longrightarrow T^{(1)}$. Consider the short exact sequences

$$\alpha_*\varepsilon: \quad 0 \longrightarrow T^{(N-1)} \longrightarrow \alpha_* M \xrightarrow{j_\alpha} M_2 \longrightarrow 0$$

and

$$\beta^*\eta_N: \quad 0 \longrightarrow T^{(N-1)} \longrightarrow \beta^* T^{(N)} \xrightarrow{j_\beta} M_2 \longrightarrow 0$$

where the second sequence is obtained via β from the standard short exact sequence

$$\eta_N: \quad 0 \longrightarrow T^{(N-1)} \xrightarrow{i_N} T^{(N)} \xrightarrow{j_{N1}} T^{(1)} \longrightarrow 0$$

Notice that there is a pure embedding $M \longrightarrow \alpha_* M$. Notice also that in the both sequences $\alpha_*\varepsilon$ and $\beta^*\eta_N$, the epimorphic maps j_α and j_β are induced by multiplication by $[\pi_0^{N-1}]$. In other words, the endomorphism $[\pi_0^{N-1}]$ on $\alpha_* M$ and, resp., on $\beta^* T^{(N)}$ is a composition of j_α and, resp., of j_β with the natural inclusion of M_2 into $\alpha_* M$ and, resp., into $\beta^* T^{(N)}$ (which is induced by the embedding $M_2 \subset T^{(1)}$).

Let

$$0 \longrightarrow T^{(N-1)} \longrightarrow \widetilde{M} \longrightarrow M_2 \longrightarrow 0$$

be the difference of $\alpha_*\varepsilon$ and $\beta^*\eta_N$ considered as elements of the group $\mathrm{Ext}_{\mathrm{Mod}'(O_0)_O}(M_2, T^{(N-1)})$. It is easy to see that $[\pi_0^{N-1}](\widetilde{M}) = 0$ and there is an $\widetilde{M}_1 \in \mathrm{Mod}'(O_0)_O$, a surjective map $\tilde{\jmath}$ and a pure embedding $\tilde{\imath}$ such that

$$\beta^* T^{(N)} \oplus \widetilde{M} \xrightarrow{\tilde{\jmath}} \widetilde{M}_1 \xleftarrow{\tilde{\imath}} \alpha_*(M)$$

By inductive assumption, there is a pure embedding of $\beta^* T^{(N)} \oplus \widetilde{M}$ into a π_0-divisible group in the category $\mathrm{Mod}'(O_0)_O$. This implies easily the existence of such an embedding for $\alpha_* M$ and, therefore, for M.

Theorem 3 is proved.

3.5. *Relation to the mixed characteristic case.*

In nn.3.5.1-3.5.3 below we need a full subcategory $\mathrm{FGr}'_1(O_0)_O$ in $\mathrm{FGr}'(O_))_O$. It consists of $\mathcal{G} = (G, G^\flat)$ such that $[\pi_0]\,\mathrm{id}_G = 0$. Then the functor L induces an antiequivalence of this category and the full subcategory $\mathrm{Mod}'_1(O_0)_O$ of $\mathrm{Mod}'(O_0)_O$ consisting of free O-modules M such that $\pi_0 M \subset \Phi(M_\sigma) \subset M$. As usually, $e = e(K/K_0)$ is the ramification index of $K = \mathrm{Frac}\, O$ over $K_0 = \mathrm{Frac}\, O_0$.

3.5.1. Suppose $e(K/K_0) = 1$.

Introduce the category $\mathrm{SH}_1(\mathbb{F}_q)_O$ with objects $(M^0, M^1, \varphi_0, \varphi_1)$, where M^0 is an O-module of finite length such that $\pi_0 M = 0$, $\varphi_0 : M^0 \longrightarrow M^0$ is a σ-linear morphism, $M^1 = \operatorname{Ker}\varphi_0$, $\varphi_1 : M^1 \longrightarrow M^0$ is a σ-linear morphism and $\varphi_0(M^0) + \varphi_1(M^1) = M$. This is an analogue of Fontaine's category of filtered modules.

Consider the functor $\mathrm{SH} : \mathrm{Mod}_1'(O_0)_O \longrightarrow \mathrm{SH}_1(\mathbb{F}_q)$ defined by the correspondence $M \mapsto (M^0, M^1, \varphi_0, \varphi_1)$, where $M^0 = M \bmod \pi_0 M$, $\varphi_0 = \Phi \bmod \pi_0$ and φ_1 is induced by $\dfrac{1}{\pi_0}\Phi$. The functor SH is an equivalence of categories if $q > 2$ and is "very close" to an equivalence if $q = 2$. This shows that strict O_0-modules have similar description as conventional group schemes if $e = 1$. Actually, one can develop the Dieudonne theory in the context of strict modules and realise the approach from [Ab1] to recover directly an analogue of Fontaine's classification of group schemes over Witt vectors.

3.5.2. Suppose $e \leqslant q - 1$.

Define the category $\mathrm{SH}_1(\mathbb{F}_q)_O$ of the collections $(M, M^0, M^1, \varphi_0, \varphi_1)$, where M is an O-module of finite length killed by π_0, $M^0 = \operatorname{Ker}\pi|_M$, $\varphi_0 : M^0 \longrightarrow M$ is a σ-linear map, $M^1 = \operatorname{Ker}\varphi_0$, $\varphi_1 : M^1 \longrightarrow M$ is a σ-linear map, and $\varphi_0(M^0) \otimes_k O + \varphi_1(M^1) \otimes_k O = M$. This category is an analogue of the category SH_O from [Ab3]. Consider the functor $\mathrm{SH} : \mathrm{Mod}_1'(O_0)_O \longrightarrow \mathrm{SH}(\mathbb{F}_q)_O$ defined by the correspondence $M \mapsto (\bar{M}, \bar{M}^0, \bar{M}^1, \varphi_0, \varphi_1)$, where $\bar{M}$ is the O-submodule in $\dfrac{\pi}{\pi_0} O \otimes M \bmod \pi M$ generated by the images of the elements of the sets $\{\dfrac{\pi}{\pi_0}\Phi(m \mid m \in M\}$ and $\{\dfrac{1}{\pi_0}\Phi(m) \mid m \in M,\ \Phi(m) = 0\}$. Then $\bar{M}^0 = \operatorname{Ker}\pi|_{\bar{M}} = M \bmod \pi M$, $\varphi_0 : \bar{M}^0 \longrightarrow M$ is induced by $\dfrac{\pi}{\pi_0}\Phi$, $\bar{M}^1 = \operatorname{Ker}\varphi_0$ and $\varphi_1 : \bar{M}^1 \longrightarrow M$ is induced by $\dfrac{1}{\pi_0}\Phi$.

If $e < q - 1$ then SH is an equivalence of categories and, if $e = q - 1$ then SH is "very close" to an equivalence of categories. Again, the methods from [Ab3] can be used to obtain directly the classification of objects from $\mathrm{FGr}_1'(O_0)_O$ via objects of the category $\mathrm{SH}(\mathbb{F}_q)_O$. (The details will appear in W.Gibbons's thesis.)

Notice that when working with equal characteristic case the category $\mathrm{SH}(\mathbb{F}_q)_O$ can be replaced by a simpler category $\mathrm{SH}'(\mathbb{F}_q)_O$ consisting of the triples $(\bar{M}, \bar{M}^0, \varphi)$, where $\bar{M}$ is an O-module of finite length killed by π_0, $\bar{M}^0 = \operatorname{Ker}\pi|_{\bar{M}}$ and $\varphi_0 : \bar{M}^0 \longrightarrow \bar{M}$ is a σ-linear morphism such that $\varphi(\bar{M}^0) \otimes O = \bar{M}$. The functor $\mathrm{SH}' : \mathrm{FGr}_1'(O_0)_O \longrightarrow \mathrm{SH}'(\mathbb{F}_q)_O$ is defined by the correspondence $M \mapsto (\bar{M}, \bar{M}^0, \varphi_0)$, where $\bar{M}$ is an O-submodule in $\dfrac{1}{\pi_0} O \otimes M \bmod \pi M$ generated by elements of the form $\dfrac{1}{\pi_0}\Phi m$ with $m \in M$, $\bar{M}^0 = \operatorname{Ker}\pi|_M = M \bmod \pi M$ and φ_0 is induced by $\dfrac{1}{\pi_0}\Phi$ (this map is still σ-linear because of the equal characteristic situation).

3.5.3. Suppose that $e(K/K_0)$ is arbitrary.

Introduce the category $\mathrm{BR}_1(\mathbb{F}_q)_O$. Its objects are the triples (M, M^1, φ), where M is an O-module of finite length such that $\pi_0 M = 0$, M^1 is an O_1-submodule in $M \otimes_O O_1$, where $O_1 = O[\pi_1]$ with $\pi_1^p = \pi$, and $\varphi : M^1 \longrightarrow M$ is a σ-linear map (σ is still the q-th power map) such that $\varphi(M^1) = M$. This category is an equicharacteristic version of Breuil's category $\mathrm{Mod}_{/S_1}$ appeared in his classification of period p group schemes in the mixed characteristic case, cf. [Br].

Again there is a natural functor from $\mathrm{Mod}'_1(O_0)_O$ to $\mathrm{BR}_1(\mathbb{F}_q)_O$ defined by the correspondence $M \mapsto (\bar{M}, \bar{M}^1, \varphi)$, where $\bar{M} = M \bmod \pi_0$ and φ is induced by $\dfrac{1}{\pi_0}\Phi$. But one can't expect that Breuil's method works in the equal characteristic case, because it is based very heavily on crystalline technique.

3.5.4. Again $e(K/K_0)$ is arbitrary.

Introduce an equal characteristic analogue of the concept of p-etale φ-module of q-heighjt 1 over S from [Fo4].

First, introduce 2nd copy $\underline{O}_0 = \mathbb{F}_q[[\underline{\pi}_0]]$ of O_0 (actually, $\underline{\pi}_0$ is an analogue of $[\pi_0]$). Then $\underline{O}_0$ will be considered as an analogue of $\mathbb{Z}_p$ and $\widetilde{\underline{O}}_0 = k[[\underline{\pi}_0]]$ is an analogue of Witt vectors $W(k)$. Then consider $S = \widetilde{\underline{O}}_0[[\pi]]$, where $k[[\pi]] = O$ appears as a subring of S. Introduce $\sigma : S \longrightarrow S$ such that $\sigma|_{\underline{O}_0} = \mathrm{id}$ and $\sigma|_O$ is (as earlier) the qth power map.

Suppose that M is an S-module of finite type together with an $\underline{O}_0$-linear map $\Phi : M_\sigma := M \otimes_{(O,\sigma)} O \longrightarrow M$. Then M is called a $\underline{\pi}_0$-etale φ-module of q-height 1 (with $q = \underline{\pi}_0 - \pi_0$), if $\mathrm{Coker}\,\Phi$ is killed by multiplication by $\underline{\pi}_0 - \pi_0$.

With this notation Theorem 2 establishes a description of the category of finite flat strict O_0-modules over O in terms of $\underline{\pi}_0$-torsion $\underline{\pi}_0$-etale φ-modules of q-height 1.

4. Properties of arising Galois modules.

In this section $\mathcal{G} = (G, G^\flat) \in \mathrm{FGr}'(O_0)_O$, $H = G(K_{\mathrm{sep}})$ is $O_0[\Gamma_K]$-module of geometric points of G and $e = e(K/K_0)$ — is the ramification index of K over K_0. We also set $\Gamma_K = \mathrm{Gal}(K_{\mathrm{sep}}/K)$ and denote by I_K the inertia subgroup of Γ_K.

4.1. *Characters of the semisimple envelope of H.*

Suppose $\bar{k}$ is an algebraic closure of k and the character $\chi : I_K \longrightarrow \bar{k}^*$ appears with a nonzero multiplicity in the semisimple envelope of the $O_0[\Gamma_K]$-module H. An analogue of the Serre Conjecture for H can be stated as follows.

Theorem 4. *For the above character χ, there are $a, N \in \mathbb{N}\setminus p\mathbb{N}$ such that $\chi = \chi_N^a$, where $a = a_0 + a_1 q + \dots + a_{N-1}q^{N-1}$ with $0 \leqslant a_i \leqslant e$ and $\chi_N : I_K \longrightarrow \bar{k}^*$ is such that for any $\tau \in I_K$, $\chi_N(\tau) = \tau(\pi_N)/\pi_N$, where $\pi_N \in K_{\mathrm{sep}}$ and $\pi_N^{q^N-1} = \pi$.*

Proof. This can be deduced in the same way as it has been obtained in the case of usual group schemes in [Ra]. First we can assume that $k = \bar{k}$ and $e < q - 1$. Then any simple object of the category $\mathrm{Mod}'(O_0)_O$ appears in the form $M = \oplus_{0\leqslant i<n} O m_i$, where $\Phi m_0 = \pi^{a_0} m_1, \dots, \Phi m_{N-1} = \pi^{a_{N-1}} m_0$ and $[\pi_0]m_0 = \dots = [\pi_0]m_{N-1} = 0$ with $0 \leqslant a_i \leqslant e$, $0 \leqslant i < N$.

Then the corresponding Galois module consists of K_{sep}-points of the O-algebra $O[T_0, \dots, T_{N-1}]$, where $T_0^q = \pi^{a_0} T_1, \dots, T_{N-1}^q = \pi^{a_{N-1}} T_0$. It can be naturally identified with the $O_0[\Gamma_K]$-module $\{\alpha \pi_N^a \mid \alpha \in \mathbb{F}_{q^N}\}$, where $a = a_0 + a_1 q + \dots + a_{N-1}q^{N-1}$. Clearly, I_K acts on it via the conjugacy class of characters $\{\sigma^i \chi_N^a \mid 0 \leqslant i < N\}$.

Remark. Following Raynaud's method one can deduce from the above description of simple objects in $\mathrm{Mod}'(O_0)_O$ that if $e < q - 1$ then the functor $G \mapsto G(K_{\mathrm{sep}})$ is a fully faithful functor from $\mathrm{Mod}'(O_))_O$ to the category of $O_0[\Gamma_K]$-modules.

4.2. *Ramification estimates.*

These estimates are given in Theorem 5 below and are completely similar to the known estimates in the case of conventional group schemes, cf. [Fo2]. The proof is based on the knowledge of "equations" of the strict module G and is done below by the methods of the paper [Ab4]. Notice that the methods from [Fo3] also can be adjusted to obtain the same estimates.

Theorem 5. *If H is killed by $[\pi_0^N]$ then the ramification subgroups $\Gamma_K^{(v)}$ act trivially on H for $v > e\left(N + \dfrac{1}{q-1}\right) - 1$.*

Proof. We can assume that there is a π_0-divisible group $\{\mathcal{G}^{(i)}\}_{i\geqslant 1}$ of a height h in $\mathrm{FGr}'(O_0)_O$ such that $\mathcal{G}^{(N)} = \mathcal{G}$.

4.2.1. By the above assumption, $L(\mathcal{G})$ is a free O-module of rank hN and we can choose its O-basis in the form

$$m_1, \dots, m_h, [\pi_0]m_1, \dots, [\pi_0]m_h, \dots, [\pi_0]^{N-1}m_1, \dots, [\pi_0^{N-1}]m_h$$

Introduce vector-columns

$$\bar{m}_1 = (m_1, \dots, m_h)^t, \dots, \bar{m}_N = [\pi_0^{N-1}]\bar{m}_1 = ([\pi_0^{N-1}]m_1, \dots, [\pi_0^{N-1}]m_h)^t$$

Then, in the obvious notation, the structure of $M = L(\mathcal{G})$ can be given in the following form

$$\frac{1}{\pi_0}C_1\Phi\bar{m}_N + \frac{1}{\pi_0}C_2\Phi\bar{m}_{N-1} + \dots + \frac{1}{\pi_0}C_N\Phi\bar{m}_1 = \bar{m}_N - \frac{1}{\pi_0}\bar{m}_{N-1}$$

$$\frac{1}{\pi_0}C_1\Phi\bar{m}_{N-1} + \dots + \frac{1}{\pi_0}C_{N-1}\Phi\bar{m}_1 = \bar{m}_{N-1} - \frac{1}{\pi_0}\bar{m}_{N-2}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\frac{1}{\pi_0}C_1\Phi\bar{m}_1 = \bar{m}_1$$

where all $C_i \in \mathrm{GL}(h, O)$ and $\det C_1 \neq 0$.

Consider vector columns $\bar{X}_i = (X_{i1}, \dots, X_{ih})^t$ of independent variables X_{ij}, $1 \leqslant i \leqslant N$, $1 \leqslant j \leqslant h$. Then the algebra $A(G)$ appears as a quotient of $O[\bar{X}_1, \dots, \bar{X}_N]$ by the ideal generated by equations

$$\sum_{1\leqslant i\leqslant s} C_i\bar{X}_{s+1-i}^q = \pi_0\bar{X}_s - \bar{X}_{s-1} \tag{5}$$

where $1 \leqslant s \leqslant N$ and by definition $\bar{X}_0 = \bar{0}$.

Consider the points of $G(K_{\mathrm{sep}})$ as solutions $\bar{a} = (\bar{a}_1, \dots, \bar{a}_N)$ of the system (5). The following lemma can be easily proved by induction on N.

Lemma 5. *If $\bar{a} = (\bar{a}_1, \dots, \bar{a}_N)$ and $\bar{a}' = (\bar{a}'_1, \dots, \bar{a}'_N)$ are solutions of (4.1) such that $\bar{a} \equiv \bar{a}' \bmod \pi_0^{\frac{1}{q-1}} m_{\mathrm{sep}}$, where m_{sep} is the maximal ideal of the valuation ring of K_{sep}, then $\bar{a} = \bar{a}'$.*

4.2.2. Suppose $\alpha \in \mathbb{Q}_{>0}$ has the p-adic valuation 0. Then $\alpha = \dfrac{m}{q^M - 1}$ with suitable $m, M \in \mathbb{N}$, $(m,p) = 1$. For any such α there is an extension K_α of K with $[K_\alpha : K] = q^M$ and the Herbrand function

$$\varphi_{K_\alpha/K}(x) = \begin{cases} x, & \text{for } 0 \leqslant x \leqslant \alpha \\ \alpha + \dfrac{x-\alpha}{q^M}, & \text{for } x \geqslant \alpha. \end{cases}$$

Notice that $\varphi_{K_\alpha/K}$ has only one edge point in $x = \alpha$.

Explicit construction of K_α can be found in [Ab4], where the field K_α appears as an extension of K of degree q^M in $L_\alpha = K(\pi_M)(T)$, where $\pi_M^{q^M-1} = \pi$ and $T^{q^M} - T = \pi_M^{-m}$. Clearly, K_α is totaly ramified over K and, therefore, there is a field isomorphism

$$h_\alpha : K \longrightarrow K_\alpha$$

From the above construction of K_α, it follows easily that h_α can be chosen in such a way that for any $a \in m_K$ (m_K is the maximal ideal in O),

$$a = h_\alpha(a)^{q^M} + \tilde{a},$$

with $\tilde{a} \in K_\alpha$ such that $v_K(\tilde{a}) \geqslant v_K(a) + \alpha$ (v_K is the normalized valuation in K).

4.2.3. Denote by the same symbol an extension of h_α to an isomorphism of K_{sep} onto $K_{\alpha,\mathrm{sep}} = K_{\mathrm{sep}}$. Clearly, $\bar{X} = (\bar{X}_1, \dots, \bar{X}_s) \mapsto h_\alpha(\bar{X}) = (h_\alpha(\bar{X}_1), \dots, h_\alpha(\bar{X}_s))$ is a one-one correspondence between solutions of the system (5) and solutions $\bar{Y} = (\bar{Y}_1, \dots, \bar{Y}_s)$ of the similar system

$$\sum_{1 \leqslant i \leqslant s} h_\alpha(C_i)\bar{Y}_{s+1-i}^q = h_\alpha(\pi_0)\bar{Y}_s - \bar{Y}_{s-1} \tag{6}$$

where $1 \leqslant s \leqslant N$ and by definition $\bar{Y}_0 = \bar{0}$.

Lemma 6. *If $\alpha > e\left(N + \dfrac{1}{q-1}\right) - 1$, then for any solution $\bar{X}^{(0)}$ of (5) there is a unique solution $\bar{Y}^{(0)}$ of (6) such that $\bar{X}^{(0)} \equiv \bar{Y}^{(0)q^M} \bmod \pi_0^{\frac{1}{q-1}} m_{\mathrm{sep}}$*

Proof of lemma. the correspondence $\bar{Y} \mapsto \bar{Z} = \bar{Y}^{q^M} - \bar{X}^{(0)}$ establishes a one-one correspondence between solutions $\bar{Y}$ of (5) and solutions $\bar{Z} = (\bar{Z}_1, \dots, \bar{Z}_s)$ of the system of equations

$$\sum_{1 \leqslant i \leqslant s} C_i \bar{Z}_{s+1-i}^q = \pi_0 \bar{Z}_s - \bar{Z}_{s-1} + \bar{F}_s \tag{7}$$

where $1 \leqslant s \leqslant N$, $\bar{Z}_0 = 0$ and

$$\bar{F}_s = \tilde{\pi}_0 \bar{Y}_s^{q^M} - \sum_{1 \leqslant i \leqslant s} \tilde{C}_i \bar{Y}_{s+1-i}^{q^{M+1}} \in \pi_0^{1+\frac{1}{q-1}} m_{\mathrm{sep}}$$

because $v_K(\tilde{\pi}_0), v_K(\tilde{C}_i) \geqslant 1 + \alpha > e\left(1 + \dfrac{1}{q-1}\right)$.

Now induction on s shows that the system (7) has a unique solution $\bar{Z}$ with coordinates in $\pi_0^{\frac{1}{q-1}} m_{\mathrm{sep}}$.

Lemma is proved.

With the above notation and assumptions we have the following corollary.

Corollary. *Suppose E, resp. E_α, is obtained by joining to K, resp. K_α, all coordinates of all solutions of the system of equations (4.1), resp. (4.2), in K_{sep}. Then $EK_\alpha = E_\alpha$.*

4.2.4. For any finite extension $F \subset L$ in K_{sep}, let $v(L/F)$ be the minimal rational number such that the ramification groups $\Gamma_F^{(v)}$ act trivially on L for $v > v(L/F)$.

Proposition 7. *With the above notation there is the following inequality*

$$v(E/K) \leqslant e\left(N + \frac{1}{q-1}\right) - 1$$

Proof. Suppose this inequality does not hold. Then there is a rational number α satisfying the assumptions from the beginning of n.4.2.2 and the inequalities

$$v(E/K) > \alpha > e\left(N + \frac{1}{q-1}\right) - 1.$$

Notice that $E_\alpha = EK_\alpha$ implies that

$$v(E_\alpha/K) = \max\{v(E/K), v(K_\alpha/K)\} = v(E/K)$$

On the other hand, looking at the maximal edge points of Herbrand functions from the identity $\varphi_{E_\alpha/K} = \varphi_{E_\alpha/K_\alpha} \circ \varphi_{K_\alpha/K}$, we obtain that

$$v(E_\alpha/K) = \max\left\{v(K_\alpha/K), \varphi_{K_\alpha/K}(v(E_\alpha/K_\alpha))\right\} =$$

$$\max\left\{\alpha, \frac{v(E/K) - \alpha}{q} + \alpha\right\} < v(E/K)$$

because $v(E/K) = v(E_\alpha/K_\alpha)$. Contradiction.

Theorem 5 is proved.

4.3. As it was noticed in n.3.5.1, if $e = 1$ then killed by $[\pi_0]$ strict O_0-modules behave very similarly to group schemes of period p over Witt vectors. For this reason, one can apply directly methods from [Ab2] to prove the following result.

Theorem 6. *Suppose H is an $\mathbb{F}_q[\Gamma_K]$-module such that*
a) *the action of inertia subgroup of Γ_K on the semisimple envelope of H is given by characters, which satisfy Serre's Conjecture, cf. Theorem 4;*
b) *the ramification subgroups $\Gamma_K^{(v)}$ act trivially on H if $v > \dfrac{1}{q-1}$ (i.e. the ramification estimate from Theorem 5 holds for H).*

Then there is an $\mathcal{G} = (G, G^\flat) \in \mathrm{FGr}'(O_0)_O$ such that $H \simeq G(K_{\mathrm{sep}})$.

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